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APPROXIMATE FORMULAS FOR THE STRESS INTENSITY FACTORS AT THE TIP OF A CRACK UNDER LONGITUDINAL SHEAR STRESS†

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AN ASYMPTOTIC form of the stress intensity factor (SIF) at the free tip of an edge crack under longitudinal shear is obtained for an arbitrary, singly connected configuration in the case when the crack is relatively short or when the distance between the end of the crack and boundary of the body is small. Analogous problems for the SIF at the apex of a corner notch is considered for simpler configurations.

1. Longitudinal shear is a homogeneous isotropic elastic body of shape ω is characterized by the displacement $W_z(x, y)$ which satisfies Laplace's equation $\nabla_{xy}{}^2 W_z = 0$ $((x, y) \in \omega)$, and the stresses $\tau_{xy} = \tau_x = \mu \partial W_z/\partial x$, $\tau_{yx} = \tau_y = \mu \partial W_z/\partial y$ (μ = const is a characteristic of the material). We shall consider, in the expanded complex plane C_z: $z = x + iy$, a singly connected polygonal region ω, for which the point $z = \infty$ lies either on, or outside the boundary. Let the load $\partial\omega$ be applied to the boundary curve $\tau_{\mathbf{n}}(\xi) = \tau_x(\xi) \sin\theta - \tau_y(\xi) \cos\theta$, where $\mathbf{n}$ is the vector of the outer normal to $\partial\omega$ and $\theta = \theta(\xi)$ is the argument of the vector of a tangent to $\partial\omega$ at its point ξ. The real-valued functions $\tau_x(\xi)$, $\tau_y(\xi)$ $(\xi \in \partial\omega)$ are continuous on $\partial\omega$. If $\infty \in \partial\omega$, then we shall require for each tip that the following condition holds:

$$\int_{\partial\omega} \ln |W(\xi) - W(\xi_k)| \, \tau_{\mathbf{n}}(\xi) \, ds < \infty$$

Here ds is the length element on $\partial\omega$ and the function $W = W(z)$ $(z \in \omega)$ maps, one-sheetedly and conformally, the region $\omega \subset C_z$ into the upper half-planes $\operatorname{Im} W > 0$ of the complex plane C_W (if $W(\xi_k) = \infty$, then this term vanishes from the previous formula). We shall also require that when $\infty \in \partial\omega$, the following condition of self-equilibrium holds:

$$\int_{\partial\omega} \tau_{\mathbf{n}}(\xi) \, ds = 0$$

Solving, under these assumptions, the Neumann problem for the function $W_z(x, y)$ harmonic in ω, we obtain the following formula for the stress field:

$$\tau_x(z) - i\tau_y(z) = \frac{1}{\pi} W'(z) \int_{\partial\omega} \frac{\tau_{\mathbf{n}}(\xi) \, ds}{W(\xi) - W(z)} \quad (z \in \omega) \tag{1.1}$$

From this we conclude, using the Christoffel–Schwartz formula [1], that the apex of corner notches and free tips of the boundary cracks are the only singularities of the stress field. Asymptotic analysis of the integral (1.1) yields

$$\begin{gathered} \tau_x(\xi_m + re^{i\varphi}) - i\tau_y(\xi + re^{i\varphi}) = -i\varkappa_0(\xi_m)(re^{i\varphi})^{p_m - 1} + \\ + (\tau_x(\xi_m) - i\tau_y(\xi_m)) + \varepsilon(r) \quad (r \to +0, \ |\varphi| < {}^1\!/_2\pi/\rho_m) \\ \varkappa_0(\xi_m) = -\frac{1}{\pi} |c_m| \int_{\partial\omega} \frac{\tau_{\mathbf{n}}(\xi) \, ds}{W(\xi) - W(\xi_m)}, \quad \varepsilon(r) = \begin{cases} O(r^{p_m - 1}) & (1 < p_m < {}^1\!/_2) \\ O(\sqrt{r}) & (p_m = {}^1\!/_2) \end{cases} \end{gathered} \tag{1.2}$$

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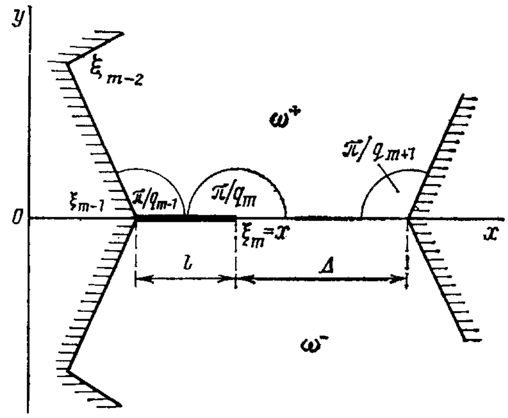

FIG. 1.

where π/p_m is the inner angle of the polygonal region ω at its tip ξ_m such, that $\frac{1}{2} \leqslant p_m < 1$, and the coefficient c_m' is determined using the relation

$$W'(z) \sim c_m (z - \xi_m)^{p_m - 1} \quad (z \in \overline{\omega}, \ z \to \xi_m) \tag{1.3}$$

The principal term of this asymptotic expression is identical, in the case of $p = p_m = \frac{1}{2}$ (i.e. in the case of a crack) with the corresponding term shown in other papers (e.g. [2, 3]).

The quantity $K_{\text{III}} = \sqrt{(2\pi\varkappa_0)}$ is called the stress intensity factor (SIF) at the singularity ξ_m of the field $\tau_x(z) - i\tau_y(z)$. From the same identity (1.1) it follows that in some neighbourhood of any finite point ξ_ν of the boundary curve with $p_\nu \geqslant 1$ the stress field is continuous in the region ω right up to the boundary.

2. Let us consider the case of a symmetrical configuration (Fig. 1). Let the configuration ω and boundary loads be symmetrical about an interval of the real axis and let the positive direction of this axis correspond to the positive direction of circumventing the region ω^+. At the same time, the real axis is the axis of the boundary crack (as in Fig. 1), or the bisectrix of the notch (provided that the cut along the segment $|T_{M-1}, \xi_M|$ in Fig. 1 is removed and we write $\xi_{m-1} = \xi_m$).

Let π/q_k be the magnitude of the inner angles of the polygonal region $\omega^+ = \omega \cap \{z: \operatorname{Im} z > 0\}$, $W = W(z)$ the function mapping uniquely and conformally the region ω^+ onto the half-plane $\operatorname{Im} W > 0$.

Using the Keldysh–Sedov method [4] to solve the mixed boundary value problem for the function $W_z(x, y)$ harmonic in ω^+, we obtain the following formula for the SIF at the free tip ξ_m of the boundary crack ($q_m = 1$) and at the apex of the corner notch ($1 < q_m < 2$):

$$K_{\text{III}}(\xi_m) = T(l) \int_{\gamma^+} \sqrt{\left| \frac{h(\xi, \xi_{m+1})}{h(\xi, \xi_m)} \right|} \, \tau_n(\xi) \, ds \quad (\xi_{m+1} \neq \infty) \tag{2.1}$$

$$K_{\text{III}}(\xi_m) = \sqrt{\frac{2q_m |c_m|}{\pi}} \int_{\gamma^+} \frac{\tau_n(\xi) \, ds}{\sqrt{h(\xi, \xi_m)}} \quad (\xi_{m+1} = \infty) \tag{2.2}$$

$$T(l) = \sqrt{\frac{2q_m |c_m|}{\pi h(\xi_{m+1}, \xi_m)}}, \quad h(x, y) = W(x) - W(y), \quad \gamma^+ = \partial\omega^+ \setminus]\xi_m, \xi_{m+1}[$$

Here c_m, q_m is the same as c_m, p_m in (1.3).

3. Let us now consider a configuration with a boundary crack (Fig. 1) and find the asymptotic expression for $K_{\text{III}}(\xi = x)$ as $l \to +0$, $l + \Delta = H = \text{const}$. In the case of a crack the function $W(z)$ (3.1) will be fixed. Using the Christoffel–Schwartz formula we obtain

$$c_m = W'(x) \sim |c_{m-1}| \, l^{q-1}, \; h(x, \xi_{m-1}) \sim q^{-1} |c_{m-1}| \, l^q \, (l \to +0), \; q = q_{m-1}$$
$$h(\xi_{m-1}, \xi) \sim q^{-1} |c_{m-1}| \, |\xi_{m-1} - \xi_m|^q \; (\xi \in]\xi_{m-2}, \xi_{m-1}[, \; \xi \to \xi_{m-1}) \tag{3.1}$$

We can assume without loss of generality that $q = q_{m-1} > 1$.

Moreover, from (3.1) we have the following relation for the first factor on the right-hand side of the formula (2.1):

$$T(l) \sim \sqrt{\frac{2|c_{m-1}|}{\pi h(\xi_{m+1}, \xi_m)}} \, l^{q-1/2} \quad (l \to +0) \tag{3.2}$$

Let us denote γ the union, with the polygonal line γ^T, of the segments not containing the tip ξ_{m-1}, and by $I(l)$ the part of the integral in (2.1) spread over γ_0. Clearly,

$$I(l) \sim \int_{\gamma_0} \sqrt{\left| \frac{h(\xi, \xi_{m+1})}{h(\xi, \xi_{m-1})} \right|} \, \tau_n(\xi) \, ds \quad (l \to +0)$$

Next we determine the contribution to the asymptotic expression being determined, of the integrals $I_{m-1}(l)$, $I_{m-2}(l)$ extending to the sides $[\xi_{m-1}, \xi_m]$ and $[\xi_{m-2}, \xi_{m-1}]$, respectively. Putting

$$\tau_n^- = \lim_{\xi \to \xi_{m-1}+0} \tau_n(\xi + i0) = -\tau_y(\xi_{m-1})$$

we obtain, from the definition of $I_{m-1}(l)$ and relations (3.1),

$$I_{m-1}(l) \sim \sqrt{h} \tau_n^- \int_{\xi_{m-1}}^{\xi_{m-1}+l} \frac{d\xi}{\sqrt{|c_{m-1}|/q} \, (l^q - (\xi - \xi_{m-1})^q)^{1/2}} \sim$$
$$\sim \tau_n^- \sqrt{\frac{hq}{|c_{m-1}|}} \, l^{1-q/2} \frac{1}{q} \int_0^1 s^{1/q-1} (1-s)^{-1/2} \, ds = -\tau_y(\xi_{m-1}) \times$$
$$\times \sqrt{\frac{h}{q|c_{m-1}|}} \, \mathrm{B}\left(\frac{1}{2}, \frac{1}{q}\right) \quad (l \to +0), \quad \mathrm{B}\left(\frac{1}{2}, \frac{1}{q}\right) \frac{\sqrt{\pi} \Gamma(1/q)}{\Gamma(1/2 + 1/q)} \tag{3.3}$$

It remains to determine the asymptotic expression for the integral

$$I_{m-2}(l) = \int_{\xi_{m-2}}^{\xi_{m-1}} \sqrt{\frac{h + h(\xi_{m-1}, \xi)}{h(x, \xi_{m+1}) + h(\xi_{m-1}, \xi)}} \, \tau_n(\xi) \, ds$$

Here we must consider three cases. If $q_{m-1} = q > 0$, then we have a node, as $l \to +0$, of the asymptotic $\xi = \xi_{m-1}$. This means that

$$I_{m-2}(l) \sim \tau_n^+ \sqrt{\frac{hq}{|c_{m-1}|}} \int_0^\delta \frac{ds}{\sqrt{l^q + s^q}} \sim \tau_n^+ \sqrt{\frac{h}{q|c_{m-1}|}} \, \mathrm{B}\left(\frac{1}{2}, \frac{1}{q}\right) \times \tag{3.4}$$
$$\times \left(\cos\frac{\pi}{q}\right)^{-1} l^{1-q/2} \quad (l \to +0)$$

$$\tau_n^+ = \lim_{\xi \to 0, \; \xi \in]\xi_{m-2}, \xi_{m-1}[} \tau_n(\xi) = -\tau_x(\xi_{m-1}) \sin\frac{\pi}{q} + \tau_y(\xi_{m-1}) \cos\frac{\pi}{q}, \quad (q = q_{m-2} > 2)$$

It is also clear that when $l \to +0$

$$I_{m-2}(l) \sim \int_{\xi_{m-2}}^{\xi_{m-1}} \sqrt{\frac{h(\xi_{m+1}, \xi)}{h(\xi_{m-1}, \xi)}} \, \tau_n(\xi) \, ds, \quad 1 < q_{m-1} < 2 \tag{3.5}$$

$$I_{m-2}(l) \sim (-\tau_x(\xi_{m-1}))\sqrt{\frac{2h}{|c_{m-1}|}}\ln\frac{h}{l}, \quad q = q_{m-1} = 2 \tag{3.6}$$

Now, from the identity $K_{\text{III}}(\xi_m) = T(l)\,(I(l) + I_{m-2}(l) + I_{m-1}(l))$ we obtain, with the aid of the asymptotic formulas (3.2)–(3.6), the following.

Remark 1. If the edge crack due to longitudinal shear makes with adjacent sides acute angles of magnitude π/q, then

$$K_{\text{III}}(\xi) \sim \left(-\tau_x(\xi_{m-1})\sqrt{\frac{2}{q}}\frac{\Gamma(1/q)}{\Gamma(1/2 + 1/q)}\operatorname{tg}\frac{\pi}{q}\right)\sqrt{l} \quad (l \to +0) \tag{3.7}$$

Remark 2. If the above angles are obtuse, then

$$K_{\text{III}}(\xi_m) \sim q^{-1/2} l^{(q-1)/2} K^*_{\text{III}}(\xi_{m-1}) \quad (l \to +0) \tag{3.8}$$

where we use as $K_{\text{III}}(\xi_{m-1})$, in accordance with (2.1), the SIF at the tip ξ_{m-1} of the corner notch of the figure ω^+ obtained from the figure ω by removing the boundary check $[\xi_{m-1}, \xi_m]$.

Remark 3. If the edge crack is perpendicular to the side adjacent to it, then

$$K_{\text{III}}(\xi_m) \sim \frac{2}{\sqrt{\pi}}(-\tau_x(\xi_{m-1}))\sqrt{l}\ln\frac{H}{l} \quad (l \to +0) \tag{3.9}$$

We note that the special case of Remark 3 when the side of the figure ω adjacent to the crack and perpendicular to it is free from external loads. Without writing out the obvious general formula, we shall consider the configuration (Fig. 2) in which we have

$$-\tau_x(0, y) = 0, \quad \tau_y(x, \pm 0) = \tau(x), \quad \tau_x(H - 0, y) = \varphi(y)$$

in the corresponding intervals, and the last function is even.

According to formula (2.1) where $\xi = \infty$, $\xi_2 = 0$, $\xi_3 = l$, $\xi = H$, if we put e.g. $W(z) = -\cos(\pi z/H)$, we obtain

$$K_{\text{III}}(l) = \sqrt{\frac{2}{H}\operatorname{tg} L}\left(\int_0^l \sqrt{\frac{1 + \cos X}{\cos X - \cos L}}\,\tau(x)\,dx + \int_0^\infty \sqrt{\frac{\operatorname{ch} Y - 1}{\operatorname{ch} Y + \cos L}}\,\varphi(y)\,dy\right. \tag{3.10}$$

$$(L = \pi l/H, \; X = \pi x/H, \; Y = \pi y/H)$$

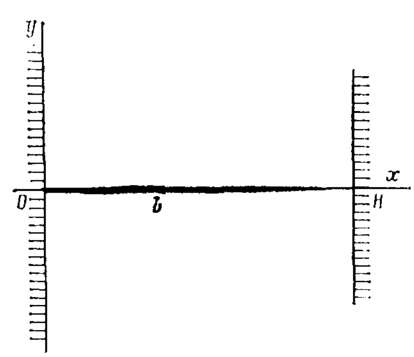

FIG. 2.

(an equivalent formula for $\varphi(y)=0$, $-\infty<y<\infty$ is given in [5]).

From this we find

$$K_{\mathrm{III}}(l)=2\sqrt{\frac{l}{\pi}}\int_0^l \frac{\tau(x)\,dx}{\sqrt{l^2-x^2}}+O\left(\frac{1}{H^2}\right)\quad (H\to\infty,\ l=\mathrm{const}) \tag{3.11}$$

The first term on the right-hand side corresponds to the SIF at the free tip of the edge crack perpendicular to the boundary of the half-plane. In particular, when $T(x)=T=\mathrm{const}$, we have the well-known formula $K_{\mathrm{III}}(l)=T\sqrt{(\pi l)}$. This means that the well-known "macroscopy principle" is satisfied in the limiting process $l=\mathrm{const}$, $H\to+0$. On the other hand, let us consider, in the same configuration, the limit $l\to+0$, $H=\mathrm{const}$ (as above the expression appearing in the general scheme $l/H\to+0$). We see that

$$K_{\mathrm{III}}(l)\sim\sqrt{\pi l}\left(\tau+\frac{1}{H}\int_0^\infty \mathrm{th}\frac{Y}{2}\,\varphi(y)\,dy\right)\quad (l\to+0),\quad H=\mathrm{const},\ \tau=\tau(+0)$$

The form of this asymptotic expression does not fully correspond to the principle under discussion, since the load $T_x(H-0,y)=\varphi(y)$ also makes a significant contribution to $K_{\mathrm{III}}(l)$. It is only on making an additional limiting passage $H\to+\infty$ that we obtain the formula for a half-plane with an edge crack perpendicular to its boundary.

4. Let us consider a configuration with an edge crack (Fig. 1) and find the asymptotic expression for $K_{\mathrm{III}}(\xi_m=x)$ when $\Delta\to+0$, $H=\mathrm{const}$. To do this we first continue to function $\tau_n(x)$ from the segment $[\xi_{m-1},\xi_m]$ to the interval $[\xi_{m-1},\xi_{m+1}]$, while retaining the continuity and absolute integrability. From relation (2.1) we obtain (see also [7, 8])

$$K_{\mathrm{III}}(\xi_m)\sim\sqrt{\frac{2q_{m-1}}{\pi\Delta}}\,Q\quad(\Delta\to+0),\quad H=\mathrm{const},\quad Q=\int_{\partial\omega+}\tau_n(\xi)\,d\xi \tag{4.1}$$

We note that if ξ_{m+1} is the tip of a coaxial crack, then $q_{m+1}=\frac{1}{2}$ and we shall have $q_{m+1}=1$ for the point ξ_{m+1} on the free surface perpendicular to the initial crack. We find, within the framework of the model under discussion, that when internal dislocations are present, fracture is somewhat stronger on the outside than on the inside [although the order of the quantity $K_{\mathrm{III}}\approx\Delta^{-1/2}(\Delta\to+0)$ is the same in both cases]. In other words, the nearness of the boundary surface induces fracture more strongly than the nearness of an elastic singularity.

5. We shall use much simpler models to deal with analogous problems for a corner notch (Figs 3 and 4).

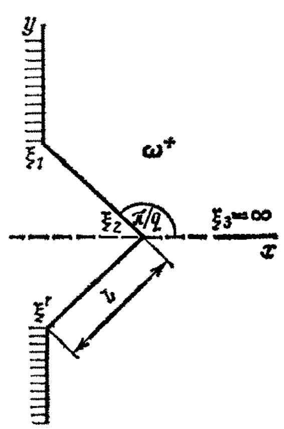

FIG. 3.

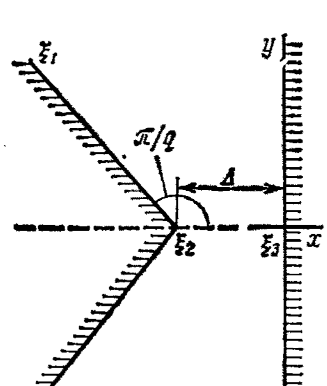

FIG. 4.

In Fig. 3 we assume that only the edges of the notch are under a constant load $\tau(\xi) = \tau = \text{const}$. Making in formula (3.1) the change of the variable of integration $\xi = W(u)$ and using the above information about the load, we obtain

$$K_{\text{III}}(\xi_2) = r\sqrt{2/\pi}\,(q\,|F_2|)^{1-q/2} \int_{u_1}^{u_2} \frac{|\xi'(u)|\,du}{|F_2|\sqrt{|u_2-u|}} \tag{5.1}$$

Here u_1, u_2 are the mappings of the points ξ_1, ξ_2 under the mapping $W = W(z)$, and the coefficient F_2 is found from the relation

$$Z'(W) \sim F_2 (W - u_2)^{1/q-1} \quad (\text{Im } W \geqslant 0,\ W \to u_2)$$

Putting $z(\infty) = \infty$, $z(0) = \xi$, $z(1) = \xi_2$ we obtain $z'(W) = cW^{1/2-1/q}(W-1)^{1/q-1}$, and the equation for determining $|c|$ in the form:

$$|\xi_2 - \xi_1| = l = |\xi| \int_0^1 u^{1/2-1/q}(1-u)^{1/q-1}du$$

Hence we obtain, in accordance with formula (5.1),

$$K_{\text{III}}(\xi_2) \sim \tau\sqrt{\pi}\left|\cos\frac{\pi}{q}\right|^{-1} 2^{q-1/2}\left(\frac{q\sqrt{\pi}l}{\Gamma(3/2-1/q)\,\Gamma(1/q)}\right)^{1-q/2} (l \to +0),\quad 1 \leqslant q \leqslant 2 \tag{5.2}$$

When $q = 1$ we have, as before, $K_{\text{III}}(\xi_2) \sim \tau\sqrt{(\pi l)}$. When q approaches the value $q = 2$, the dependence on l weakens, but the coefficient of $l^{1-q/2}$ increases without limit. This corresponds to the general principle which states, that, when the singularity index increases monotonically to $+0$, the corresponding SIF increases to $+\infty$.

In Fig. 4 we can assume that all sides of the figure are loaded. Using formula (3.1) and relation

$$|c_m| = q_{q_m}^{1-q_m}\,|F_m|^{-q_m},$$

we obtain

$$K_{\text{III}}(\xi_2) \sim c(q)\,Q\Delta^{-q/2} \quad (\Delta \to 0),\quad 0 \leqslant q < 2 \tag{5.3}$$

where Q, as before, denotes the integral of the load and the coefficient $C(q)$ is given by the formula

$$C(q) = \sqrt{2/\pi}\,q^{1-q/2}\,(B(1/2,\ 1/q))^{q/2},\quad 1 \leqslant q < 2 \tag{5.4}$$

Putting $q_{m+1} = \frac{1}{2}$ in (4.1) or in $q = 1$ (5.4) we obtain for the case when the crack reaches the free surface perpendicular to it, the relation

$$K_{\text{III}} \sim Q/\sqrt{\pi\Delta} \quad (\Delta \to +0) \tag{5.5}$$

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